ANALYSIS BASED CONSTRUCTION OF A HYPERELASTIC POTENTIAL FOR COLLAGENOUS TISSUES

A Project Report

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ANANTHA NARAYANAN S

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DR. KRISHNA KANNAN

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PROJECT CERTIFICATE

This is to certify that the project titled **Analysis based construction of a hyperelastic potential for collagenous tissues**, submitted by **Anantha Narayanan S (ME16B051)**, to the Indian Institute of Technology, Madras, for the award of the degree of **Bachelor of Technology (Honours)** in **Mechanical Engineering**, is a bonafide record of the project work done by him in the Department of Mechanical Engineering, IIT Madras. The contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Prof. Krishna Kannan
Project Guide
Professor
Dept. of Mechanical Engineering
IIT Madras, Chennai 600 036

Prof. N. Ramesh Babu
Head
Dept. of Mechanical Engineering
IIT Madras, Chennai 600 036

Place: Chennai

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ABSTRACT

KEYWORDS: Hyperelasticity ; Polyconvexity; *A priori* analysis; Failure envelope

Hyperelastic constitutive models relate the deformation of highly elastic materials to stresses through a strain energy density potential. Once the potential is determined, various other quantities such as stresses can be easily calculated for various modes of deformation and boundary conditions. Hence, a major challenge in hyperelasticity is to derive potentials with the least number of parameters through molecular statistical methods or through mathematical analysis. In this project, a novel method to construct potentials through *a priori* analysis of mathematical inequalities has been proposed. In particular, the inequality of polyconvexity has been exploited to ensure the existence of solutions. The proposed model can also capture the stretch limit of deformation and thereby the failure envelope of the material. Finally, the predictions of the model are compared with biaxial tension data of small intestinal tissues to verify its accuracy. The failure envelope of the tissue for all modes of deformation are also predicted. This is of particular significance in the modelling of various medical and surgical procedures.

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CHAPTER 1

INTRODUCTION

1.1 Hyperelasticity

Modelling the mechanical behavior of materials requires constitutive equations. Certain materials such as rubbers and biological materials show elasticity beyond the usual limits observed in conventional materials such as metals. These materials are typically modelled using hyperelastic constitutive laws (Treloar (1975)).

Hyperelastic constitutive models relate the deformation of highly elastic materials to stresses through a strain energy density potential (Ogden (1997)). Once the potential is determined, various other quantities such as stresses can be easily calculated for various modes of deformation and boundary conditions.

These potentials could then be used in design applications such as mechanical modelling of medical procedures like balloon angioplasty, capsule endoscopy, plastic surgery and also in surgical simulations of various organs (Holzapfel *et al.* (2002); Bellini *et al.* (2011); Lapeer *et al.* (2010)).

1.2 Construction of potentials

The major challenge in hyperelasticity is to derive potentials with the least number of parameters which can guarantee physically reasonable behavior. Models based on molecular statistical theories and mathematical analysis are considered superior since they ensure reasonable physical behavior. Usually, the construction of potentials is carried out in three broad ways:

- **Ad-hoc construction**:
  The first approach involves the use of arbitrary functional forms chosen based on experimental data or prior experience (Fung (1967)). These functions could be of various types, such as polynomial, exponential, or power-law forms. Though useful for specific cases, this ad-hoc construction could fail to wholly capture the mechanical response of materials subject to other modes of deformation.

- **Molecular statistics basis**:

  Another widely used approach is to construct potentials through molecular statistical theories (Arruda and Boyce (1993)). These models incorporate microstructural features by considering the distribution and orientation of fibres within the material. While these models connect the macroscopic material behavior to the underlying physics, their construction is made tricky by the innate complexity of the collagen fibres which make up biological materials.

- **Mathematical analysis**:

  The final approach to construct potentials is through rigorous analysis (Prasad and Kannan (2020)). Mathematical forms of the potential function are determined through either *a priori* or *a posteriori* analysis to satisfy certain mathematical inequalities. Though challenging to construct, these forms have the advantage of being applicable to a broad class of materials and modes of deformation.

1.3 Constitutive inequalities

As mentioned in Section 1.2, potential forms satisfying certain mathematical inequalities possess considerable advantages over other forms. These mathematical inequalities, known as constitutive inequalities, restrict the functional forms the potential function can take to ensure reasonable physical behavior. Various constitutive inequalities have been proposed in the past, but only a few have withstood scrutiny over time (Truesdell (1992)).

One such constitutive inequality is that of convexity. However, convexity of potential functions is too restrictive and precludes the possibility of multiple solutions, which is commonly observed in physically occurring phenomena such as buckling.

Another inequality, strong ellipticity, is connected to the existence of real wave speeds in linear elasticity (Lurie (2012)). The conditions imposed by this inequality make it difficult to construct potentials leading to reasonable physical behavior across all deformation states since it is hard to verify if strong ellipticity has been satisfied.

These limitations make polyconvexity, as proposed in Ball (1976), increasingly relevant in the construction of potentials . Polyconvexity, a more restrictive inequality than ellipticity, automatically guarantees that the condition of ellipticity is met, thereby ensuring real wave speeds.

More importantly, polyconvexity, with some additional conditions, ensures the existence of at least one minimizer. This further proves the existence of solutions with considerable smoothness. This is of great significance in nonlinear theories where the existence of solutions for boundary-value problems are usually unknown, unlike the case of linear theories.

1.4 Mechanical behavior of biological materials

Before constructing potentials, it is necessary to understand the mechanical behavior of biological materials from experimental data.

Biological materials and soft tissues typically support large deformation and strain (Fung (2013)). These materials also show anisotropic behavior due to their complex fibrous microstructure. However, some soft tissues showcase nearly isotropic behavior.

Another common feature is the presence of a stretch limit, which arises when the collagen fibres of the tissue are fully stretched and can no longer undergo further deformation (Figure 1.1). This behavior leads to abnormally high stresses and rapid failure of the material at the stretch limit. A key point of observation is that this stretch limit varies with the mode of deformation. Hence, experimental data from various loading conditions such as uniaxial, biaxial, or pure shear is necessary to replicate the observed physical behavior of the tissue completely.

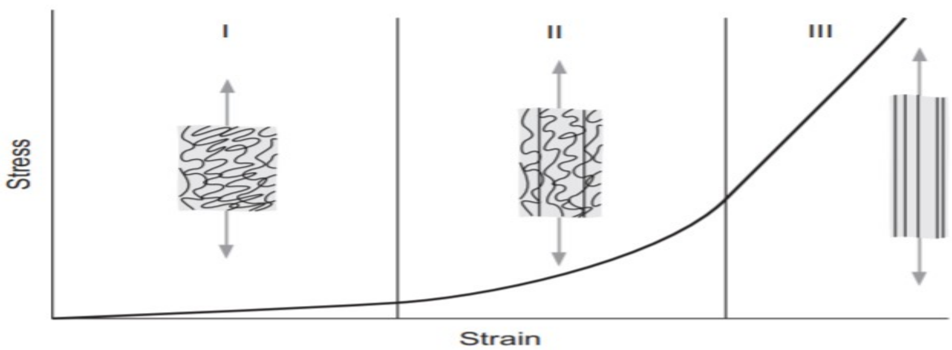

Figure 1.1: Figure shows the extension of collagen fibres present in tissues with increasing strain. Image adapted from Holzapfel and Fereidoonnezhad (2017)

CHAPTER 2

LITERATURE REVIEW

2.1 Constitutive inequalities

A brief review of various constitutive inequalities has been given in Schröder *et al.* (2005). Convexity and polyconvexity have been discussed in Ball (1976). Prasad and Kannan (2020) and Zubov and Rudev (2011) contain further information on the significance of strong ellipticity.

2.2 Constitutive models

Chagnon *et al.* (2015) and Wex *et al.* (2015) provide an in-depth review of the various hyperelastic potentials used for modelling numerous soft tissues. Some tissues can be considered to be nearly isotropic. These include spleen, liver, kidney, brain, breast tissue and lungs.

2.2.1 Polynomial and exponential type models

Porcine spleen has been modelled using exponential-type models (Davies *et al.* (1999)). Polynomial-type models have been used to model porcine liver and kidneys (Kim and Srinivasan (2005)). Umale *et al.* (2011) have used polynomial models for simulating the behavior of Glisson's capsule and hepatic veins.

Polynomial-type potential functions have been used by Raghavan and Vorp (2000) to model wall stresses in abdominal aortic aneurysms. Demiray *et al.* (1988) and Volokh and Vorp (2008) have used exponential-type functions to model abdominal aortic aneurysms.

Exponential models provide a large change in elastic modulus, a behavior observed in experimental data. Sheep brain tissue has also been modelled using exponential-type potentials (Rashid *et al.* (2014)).

2.2.2 Molecular statistics based models

A review of statistically motivated potential forms based on collagen fibre distribution has been conducted by Holzapfel *et al.* (2019). These potential forms are usually anisotropic since they incorporate complex microstructures.

One of the most popular models of this type is the Gasser-Ogden-Holzapfel model (Holzapfel *et al.* (2000)), which is used to study arterial wall mechanics. Other earlier models include those by Sacks (2003) and Lanir (1983).

2.2.3 Analysis based models

A lacunae exists in the construction of potentials through mathematical analysis. More recently, a model based on *a priori* analysis of constitutive inequalities has been proposed by Prasad and Kannan (2020) to model brain tissues.

2.2.4 Biaxial modelling

Biaxial modelling of thoracic aorta has been performed by using both isotropic and anisotropic polynomial type potentials (Ferruzzi *et al.* (2011)). Small intestinal tissues such as ileum and jejunum have also been modelled using both isotropic and anisotropic potentials (Bellini *et al.* (2011)).

Tissue	Model	Uniaxial/ Biaxial
Porcine liver (Kim and Srinivasan (2005))	Isotropic polynomial (Mooney-Rivlin)	Uniaxial
Porcine spleen (Davies *et al.* (1999))	Isotropic exponential (Fung-Demiray)	Uniaxial
Porcine kidney (Kim and Srinivasan (2005))	Isotropic polynomial (Mooney-Rivlin)	Uniaxial
Porcine hepatic vein (Umale *et al.* (2011))	Isotropic polynomial (Ogden)	Uniaxial
Arterial wall (Holzapfel *et al.* (2000))	Anisotropic molecular statistics based (GOH)	Uniaxial
Sheep brain (Rashid *et al.* (2014))	Isotropic exponential (Gent)	Uniaxial
Abdominal aorta (Ferruzzi *et al.* (2011))	Anisotropic polynomial	Biaxial
Small intestine (Bellini *et al.* (2011))	Isotropic polynomial (Mooney-Rivlin)	Biaxial

Table 2.1: Table shows the various potentials used to model biological tissues

CHAPTER 3

SCOPE

The major challenge in hyperelasticity is the construction of potentials with the fewest number of parameters based on mathematical analysis or molecular statistics. To the best of knowledge, no literature exists on the modelling of the stretch-limit and failure envelope of biological materials using constitutive inequalities.

This work aims to develop a nearly incompressible, isotropic hyperelastic potential satisfying the conditions of polyconvexity through *a priori* mathematical analysis. The potential developed must:

1. Reduce to Hooke's law upon linearization.
2. Reduce to a neo-Hookean potential under small deformation.
3. Capture the stretch limit, that is, the failure envelope of the material for all possible modes of deformation.
4. Ensure that the potential shoots to infinity at the stretch limit of deformation.
5. Contain the fewest possible number of parameters.
6. Provide the maximum possible rise in elastic modulus.
7. Reasonably simmulate the observed mechanical behavior under various deformation conditions.

To verify the predictive capabilities, the model is compared with data obtained from biaxial extension of small intestinal tissues, ileum and jejunum, under various biaxial stress ratios.

CHAPTER 4

METHOD

4.1 Kinematics and invariants

Consider a body undergoing deformation. Let $\mathbf{X}$ be a point in the reference configuration and $\mathbf{x}$ be the same point in the current configuration. Then the deformation gradient tensor, $\boldsymbol{F}$ is defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{4.1}$$

For valid physical behavior, det($\boldsymbol{F}$) must be non negative. From the theorem of polar decomposition, it can be shown that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{4.2}$$

where $\mathbf{U}$ and $\mathbf{V}$ are positive definite tensors known as the right and left stretch tensors, and $\boldsymbol{R}$ is an orthogonal rotation tensor.

The Cauchy-Green tensors are given by $\mathbf{C}$=$\mathbf{F^T}\mathbf{F}$=$\mathbf{U^2}$ and $\mathbf{C}$=$\mathbf{F}\mathbf{F^T}$=$\mathbf{V^2}$ where 'T' denotes the transpose.

The principal invariants of $\mathbf{C}$ and $\mathbf{U}$ are given by

$$\begin{aligned} I_C &= \mathrm{tr}(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ II_C &= \mathrm{tr}\left(\mathrm{cof}(\mathbf{C})\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ III_C &= \det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
i_1 &= \mathrm{tr}(\mathbf{U}) = \lambda_1 + \lambda_2 + \lambda_3 \\
i_2 &= \mathrm{tr}\left(\mathrm{cof}(\mathbf{U})\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\
i_3 &= \det(\mathbf{U}) = \lambda_1\lambda_2\lambda_3
\end{aligned}
\tag{4.4}
$$

where 'tr' denotes trace, 'det' denotes the determinant and $\lambda_1, \lambda_2, \lambda_3$, are the eigenvalues of $\mathbf{U}$.

It can also be seen that

$$
\begin{aligned}
I_C &= i_1^2 - 2\,i_2 \\
II_C &= i_2^2 - 2\,i_1\,i_3 \\
III_C &= i_3^2
\end{aligned}
\tag{4.5}
$$

In case of incompressible materials, III_C=i_3=1 since det ($\mathbf{F}$)=1 as there is no change in volume.

4.2 Model

To construct the potential, a neo-Hookean model is first considered. A neo-Hookean potential is the simplest hyperelastic model consisting of only one term.

$$
W_{\text{neo-Hookean}} = \mu_1\,(I_C - 3) \tag{4.6}
$$

where μ_1 is the modulus and and I_C is the first invariant.

It is well known that such a potential can fit elastomer data very well, but it is not well suited for biological materials since its modulus remains constant with deformation. This potential does not account for the large change in modulus and eventual failure at a stretch limit observed in soft biological materials.

To overcome these shortcomings, the neo-Hookean potential has been combined with an additional function which modifies its modulus with deformation.

$$W_{\text{modified}} = \mu_1 \, (I_C - 3) \, f(\xi) \tag{4.7}$$

where

$$\xi = \beta \, I_C + (1-\beta) \, II_C \tag{4.8}$$

β modifies the function 'f' such that it is a function of the first invariant only ($f(I_C)$) when $\beta = 1$ and function of the second invariant only ($f(II_C)$) when $\beta = 0$.

The challenge now is to identify the most suitable form of the function 'f' which will provide the maximum change in modulus and also describe the failure envelope of the material with the least number of parameters.

To achieve this, the potential function, W_{modified}, is subjected to restrictions imposed by constitutive inequalities. In particular, the inequality of polyconvexity is considered.

4.2.1 Check for polyconvexity

The following conditions to ensure polyconvexity of isotropic potentials have been postulated by Steigmann (2010):

Theorem 4.2.1 *A function W(i_1, i_2, i_3) is polyconvex if*

i. W is convex with respect to i_1,i_2 and i_3 jointly.

ii. W is a non-decreasing function of i_1 and i_2.

Theorem 4.2.1 is further exploited to arrive at the mathematical form of the potential function. The potential is assumed to be incompressible and hence it is a function of 'i_1' and 'i_2' only. Hence, the proposed potential can be expressed as follows

$$W_{\text{modified}} = \mu_1 \, (i_1^2 - 2i_2 - 3) \, f\big(\beta(i_1^2 - 2i_2) + (1-\beta) \, i_2^2 - 2i_1\big) \tag{4.9}$$

Theorem 4.2.2 *A twice differentiable function is convex in its arguments if and only if the Hessian matrix is positive definite.*

A simple check for positive definiteness is given in the following theorem:

Theorem 4.2.3 *A real symmetric matrix is positive definite if and only if all its leading principal minors are positive, that is, the determinant of all k × k upper left submatrices are positive.*

The Hessian matrix for the proposed potential is found using

$$\text{Hessian} = \begin{bmatrix} \frac{\partial^2 W}{\partial i_1^2} & \frac{\partial^2 W}{\partial i_1 \partial i_2} \\ \frac{\partial^2 W}{\partial i_1 \partial i_2} & \frac{\partial^2 W}{\partial i_2^2} \end{bmatrix} \tag{4.10}$$

For W_{modified} to be positive definite, the determinant of the hessian and the value of $\frac{\partial^2 W}{\partial i_1^2}$ must be non negative.

The resulting conditions are further simplified by considering the worst-case scenario, that is, $I_C = II_C = i_1 = i_2 = 3$.

Analysis of the determinant leads to a non-linear differential equation which on simplifying leads to the following:

$$9 - \frac{\xi^3}{3} + \frac{\log\left(\frac{a A_1 + 2(1+\sqrt{1-a}) A_2 f'(\xi)}{a A_1 - 2(-1+\sqrt{1-a}) A_2 f'(\xi)}\right)}{A_1 \sqrt{1-a}} \geq 0 \tag{4.11}$$

where

$$\begin{aligned} 0 < a < 1 \\ A_1 = \frac{12 + 3\beta}{54(9 + 16\beta^2 - 24\beta)} \\ A_2 = \frac{373\beta^2 + 292}{54(9 + 16\beta^2 - 24\beta)} \\ \text{and} f(3) = \gamma \end{aligned} \tag{4.12}$$

The parameter 'γ' controls the relative scaling of the modulus.

4.2.2 Final form of $f(\xi)$

To get the best possible functional form, the inequality is set to 0 and solved to get $f(\xi)$. Therefore, the form of the function f is found to be

$$f(\xi) = \gamma + \frac{a A_1}{2 A_2} \int_3^{\xi} \frac{1}{-1 + \sqrt{1-a}\, A_2 \coth\left(\frac{1}{6}\sqrt{1-a}\, A_1 (27 - x^3)\right)} dx \tag{4.13}$$

Hence, the potential function becomes

$$\boxed{W_{\text{modified}} = \mu_1 (I_C - 3)\left(\gamma + \frac{a A_1}{2 A_2} \int_3^{\xi} \frac{1}{-1 + \sqrt{1-a}\, A_2 \coth\left(\frac{1}{6}\sqrt{1-a}\, A_1 (27 - x^3)\right)} dx\right)} \tag{4.14}$$

The failure envelope of the material is defined by the integrand of the function $f(\xi)$. When the material fails, the stress as well as the potential shoots up to infinity.

Since the potential is now known, all other information such as stresses can be easily derived for various modes of deformation and boundary values.

It is well known that the Cauchy stress is given by

$$\sigma_z = \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_3 \frac{\partial W}{\partial \lambda_3} \tag{4.15}$$

4.3 Growth conditions

The growth condition to ensure the existence of solutions proposed by Ball (1977) is

$$W > a + b(I_C^p) \tag{4.16}$$

where b>0, p≥ 1,

It can be seen that the growth conditions are met when a=0, b= $\gamma\mu_1$ and p=1 since the neo-Hookean model has been multiplied by a function which is monotone increasing. Moreover, The proposed potential shoots up to infinity at the stretch limit.

Hence, the proposed potential functions satisfies the conditions for the existence of solutions with considerable smoothness.

4.3.1 Stress for uniaxial loading

Consider the uniaxial case first, in this case, the deformation gradient becomes

$$\mathbf{F} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix} \tag{4.17}$$

Hence, the Cauchy stress in this case is

$$\boxed{\begin{aligned} \sigma_z = 2\mu_1\Big(\lambda^2 - \frac{1}{\lambda}\Big)\Big(\gamma + \frac{a\,A_1}{2\,A_2}\int_3^{\xi} \frac{1}{-1+\sqrt{1-a}\,A_2 \coth\left(\frac{1}{6}\sqrt{1-a}\,A_1\,(27-x^3)\right)}dx\Big) \\ +2\mu_1\Big(\lambda^2 + \frac{2}{\lambda} - 3\Big)\Big(\beta\Big(\lambda^2 - \frac{1}{\lambda}\Big) + (1-\beta)(\Big(\lambda - \frac{1}{\lambda^2}\Big)\Big) \\ \Big(\frac{a\,A_1}{2\,A_2}\int_3^{\xi} \frac{1}{-1+\sqrt{1-a}\,A_2 \coth\left(\frac{1}{6}\sqrt{1-a}\,A_1\,(27-\xi^3)\right)}\Big) \end{aligned}} \tag{4.18}$$

4.3.2 Stress for biaxial loading

In case of biaxial loading, the deformation gradient becomes

$$\mathbf{F} = \begin{bmatrix} \lambda_z^2 & 0 & 0 \\ 0 & {\lambda_\theta}^2 & 0 \\ 0 & 0 & \frac{1}{\lambda_z^2\lambda_\theta^2} \end{bmatrix} \tag{4.19}$$

Hence, the Cauchy stress in this case is

$$\boxed{\begin{aligned}\sigma_{11} = 2\mu_1\left(\lambda_z^2 - \frac{1}{\lambda_z^2\lambda_\theta^2}\right)\left(\gamma + \frac{a\,A_1}{2\,A_2}\int_3^\xi \frac{1}{-1+\sqrt{1-a}\,A_2\coth\left(\frac{1}{6}\sqrt{1-a}\,A_1\,(27-x^3)\right)}\,dx\right)\\ +2\mu_1\left(\lambda_z^2+\lambda_\theta^2+\frac{1}{\lambda_z^2\lambda_\theta^2}-3\right)\left(\beta\left(\lambda_z^2-\frac{1}{\lambda_z^2\lambda_\theta^2}\right)+(1-\beta)(\left(\lambda_z^2\lambda_\theta^2-\frac{1}{\lambda_z^2}\right)\right)\\ \left(\frac{a\,A_1}{2\,A_2}\int_3^\xi \frac{1}{-1+\sqrt{1-a}\,A_2\coth\left(\frac{1}{6}\sqrt{1-a}\,A_1\,(27-\xi^3)\right)}\right)\end{aligned}} \tag{4.20}$$

$$\boxed{\begin{aligned}\sigma_{22} = 2\mu_1\left(\lambda_\theta^2 - \frac{1}{\lambda_z^2\lambda_\theta^2}\right)\left(\gamma + \frac{a\,A_1}{2\,A_2}\int_3^\xi \frac{1}{-1+\sqrt{1-a}\,A_2\coth\left(\frac{1}{6}\sqrt{1-a}\,A_1\,(27-x^3)\right)}\,dx\right)\\ +2\mu_1\left(\lambda_z^2+\lambda_\theta^2+\frac{1}{\lambda_z^2\lambda_\theta^2}-3\right)\left(\beta\left(\lambda_\theta^2-\frac{1}{\lambda_z^2\lambda_\theta^2}\right)+(1-\beta)(\left(\lambda_z^2\lambda_\theta^2-\frac{1}{\lambda_\theta^2}\right)\right)\\ \left(\frac{a\,A_1}{2\,A_2}\int_3^\xi \frac{1}{-1+\sqrt{1-a}\,A_2\coth\left(\frac{1}{6}\sqrt{1-a}\,A_1\,(27-\xi^3)\right)}\right)\end{aligned}} \tag{4.21}$$

CHAPTER 5

RESULTS AND DISCUSSION

To verify the accuracy of the model, it is compared with experimental data. In particular, biaxial extension data at various stress rations ($\sigma_{22} : \sigma_{11}$) of the ileum and jejunum which comprise the small intestine has been used for comparison (Bellini *et al.* (2011)). Both ileum and jejunum are nearly incompressible and show nearly isotropic behavior. They also show a stretch limit with a large rise in modulus, which makes them ideal candidates for the proposed model.

5.1 Parameter fitting

The 4 parameters required for the model have been identified using a '*fminsearch*' least square fit model in MATLAB. This is a derivative free method whose accuracy depends on the initial guess values chosen. To ensure the best possible initial values have been provided, a genetic algorithm has been first used to determine a reasonable estimate of the global minima. This estimate is then used to run the '*fminsearch*' least square fit optimization to ensure the best possible model fit.

The identified parameters for ileum and jejunum can be found in Table 5.1. The value of the parameter 'β' is zero for both ileum and jejunum which makes the modifier function dependent on the second invariant 'II_C' only.

5.2 Effect of parameters

The parameter 'μ_1' is similar to the modulus of a neo-Hookean model while 'γ' is a scaling factor between the neo-Hookean potential and the modifier function. The parameters 'a' and 'β' determine the failure envelope of the material.

Figure 5.1 shows the effect of the parameter a on the failure envelope for different values of β. It is clearly observed that a merely changes the maximum stretch limit

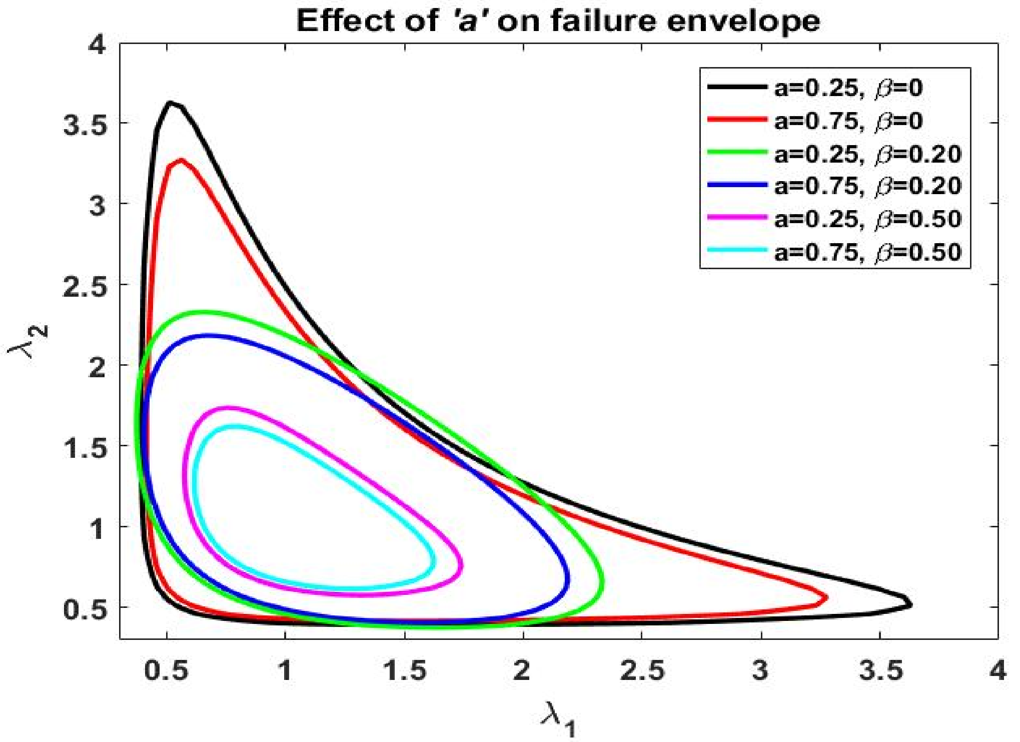

Figure 5.1: Figure shows the effect of the parameter a on the failure envelope for various values of β

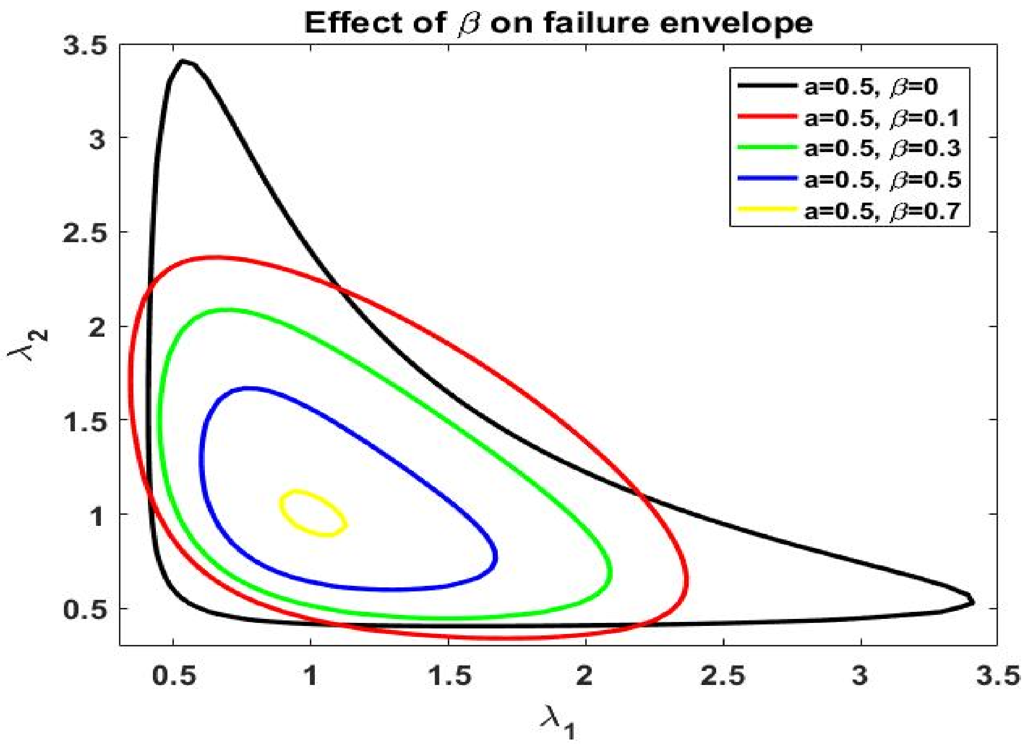

Figure 5.2: Figure shows the effect of the parameter β on the failure envelope for constant values of a

Parameter	Ileum	Jejunum
μ_1(MPa)	17.63	40.60
γ	2.01x10^{-05}	1.18x10^{-05}
a	0.7621	0.9446
β	0	0

Table 5.1: Table shows the value of parameters used to model ileum and jejunum

of the material but has no effect on the shape of the failure envelope. An increase in a decreases the stretch limit. Figure 5.2 shows the effect of β on the failure envelope for a fixed value of a. The figure shows that β affects both the shape of the failure envelope as well as the maximum possible stretch limit.

5.3 Comparison with data

Figures 5.3 and 5.5 show that the model predictions fit well with the experimental data with just 3 effective parameters a, μ_1 and γ. The dramatic increase in stress beyond a certain deformation has also been captured well by the proposed model.

Figures 5.4 and 5.6 show the failure envelope for ileum and jejunum as derived from the model using the parameters a and β. The data points obtained from biaxial extension have been denoted by black markers.

The construction of envelopes is significant because of its importance in simulation of failure in tissues during surgical and other medical procedures. However, it is not possible to accurately generate failure envelopes for materials with just uniaxial extension data. Hence, it is necessary to have at least two modes of deformation or biaxial deformation at multiple stress ratios.

The failure envelope provides the stretch combinations for all possible types of deformation where the stress shoots up to infinity and hence, the material undergoes failure. It can be seen that using just a few points of data, the entire failure envelope can be easily constructed. Another important observation is that the potential shoots up to infinity at all stretch combination on the envelope. This further ensures that the material undergoes failure.

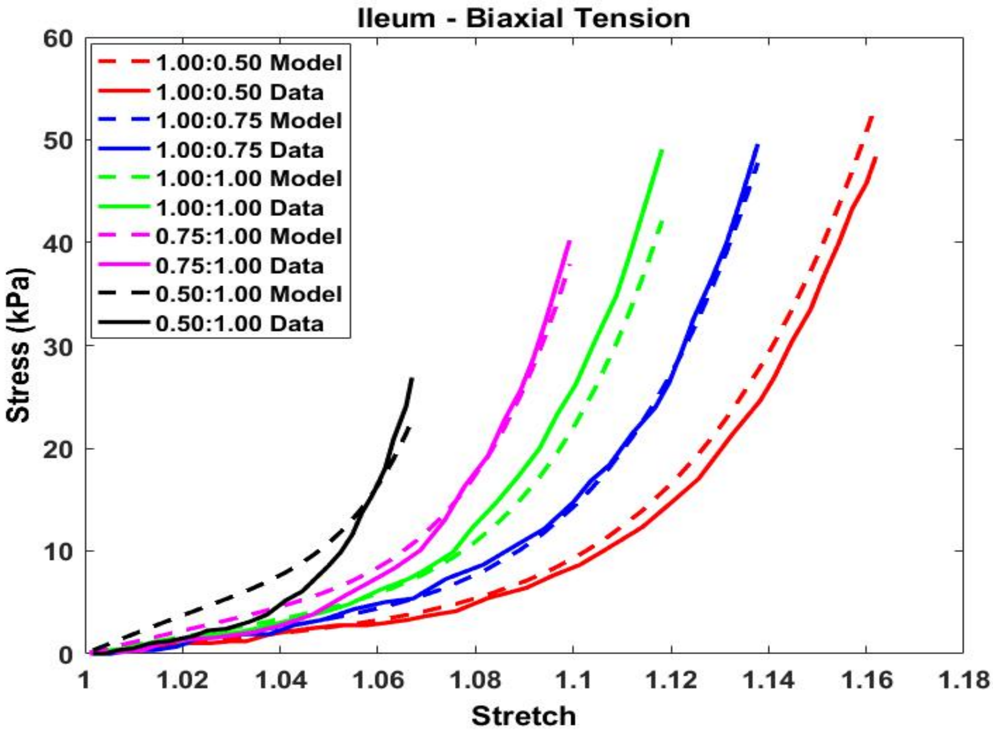

Figure 5.3: Comparison of model prediction and data for biaxial extension of ileum at various stress ratios ($\sigma_{22} : \sigma_{11}$)

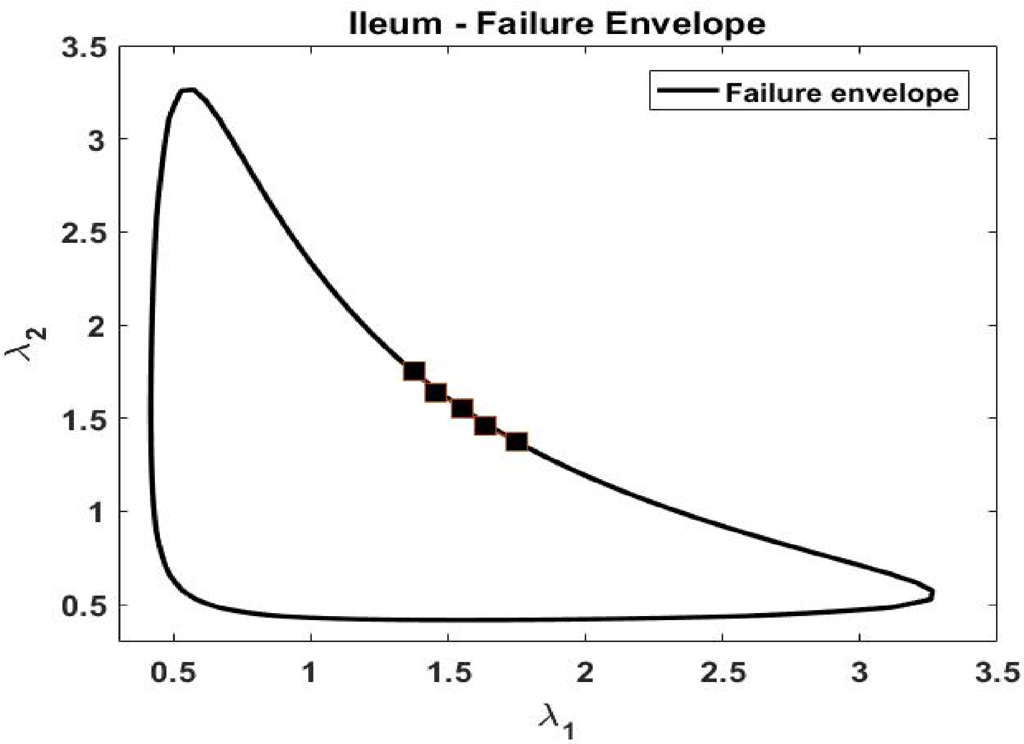

Figure 5.4: Failure envelope of ileum obtained from the model

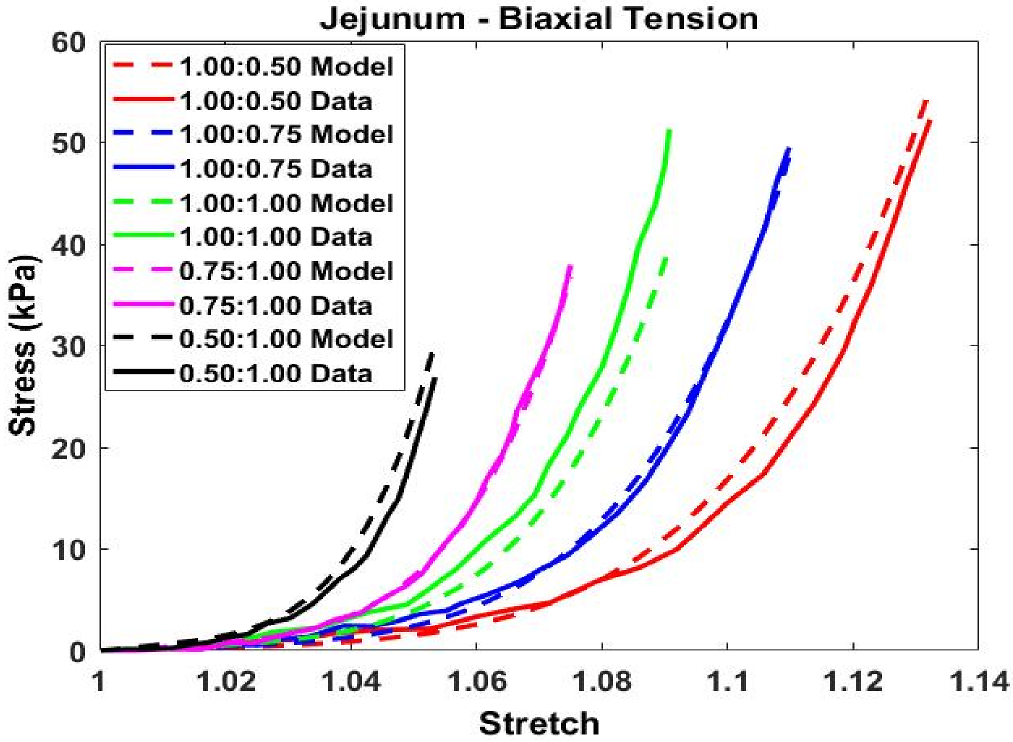

Figure 5.5: Comparison of model prediction and data for biaxial extension of jejunum at various stress ratios ($\sigma_{22} : \sigma_{11}$)

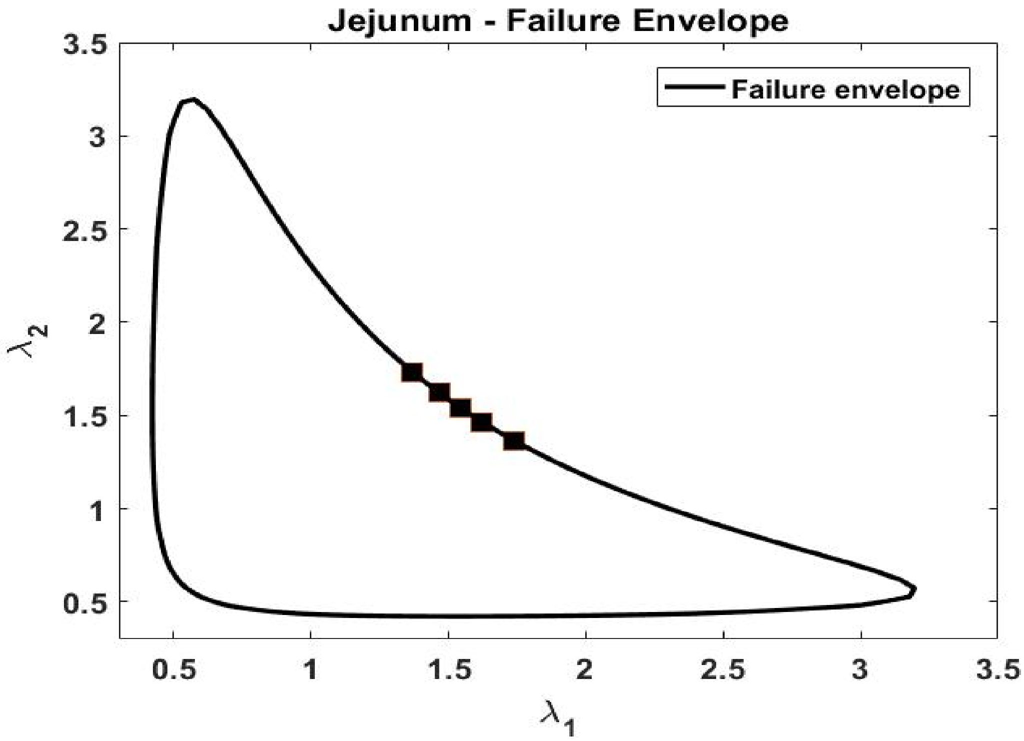

Figure 5.6: Failure envelope of jejunum obtained from the model

CHAPTER 6

CONCLUSION

One of the major challenges in hyperelasticity is the construction of potentials with the fewest number of parameters using molecular statistics or mathematical analysis. In this project, a novel method to construct potentials through *a priori* mathematical analysis of the restrictions posed by polyconvexity, a constitutive inequality, has been proposed.

Comparison with biaxial extension data of ileum and jejunum tissues from the small intestine show that the model provides good predictions with just three parameters. The model also predicts the failure envelope of the material for all possible deformation. This is of great significance in the simulation of various surgical and medical procedures.

6.1 Scope of future work

To extend the model to a wider class of materials, the following steps are proposed:

- Fit the model with other modes of deformation such as shear or compression to further validate the model when such data becomes available in the literature.
- Analyze the possibility of exponential-type solutions which can cover the entire range of deformation without a stretch limit.

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